

A NOVEL EXTENSION OF THE REDUCED-KIES FAMILY: PROPERTIES, INFERENCE, AND APPLICATIONS TO RELIABILITY ENGINEERING DATA*

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Abstract. In this work, we present a new class of continuous distributions called the exponentiated reduced-Kies-G family. The basic mathematical properties of the new family are addressed. A special sub-model of the proposed family, called the exponentiated reduced-Kies exponential (ERKiEx), is studied in detail concerning the aspect of estimation and inference, we described eight estimation methods. A comprehensive set of simulation studies are performed to compare and rank the proposed methods based on partial and overall ranks. Two real-life reliability engineering data applications are employed to explore the applicability and flexibility of the new ERKiEx distribution as compared to well-known other exponential extensions such as the Marshall–Olkin exponential, Kumaraswamy exponential and exponentiated exponential distributions.

Keywords: Reduced-Kies model, statistical modeling, T-X family, maximum likelihood estimation, maximum product of spacing, reliability data.

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1 Introduction

Statisticians have become more concerned in constructing new families of distributions by adding additional shape parameters (one or more) to a basic distribution during the past few decades. In general, a good generator's additional parameters can accept symmetric, unimodal, bimodal, right-skewed, and left-skewed density functions, as well as raise and reduce skewness and kurtosis. But perhaps most importantly, they can produce all variations of the hazard function. Additionally, the other factors can offer a lot of versatility for modelling data in a variety of fields, including economics, engineering, dependability, and the sciences related to medicine and actuarial work, among others. This work introduces a new family of continuous probability distributions. The new family is derived based on reduced Kies (RKi) (Kumar & Dharmaja, 2013) using the concept of T-X method.

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Kumar & Dharmaja (2013) proposed the RKi distribution as a special case of the Kies distribution. The cumulative distribution function (CDF) of the RKi distribution is specified by

$$F_\rho(x) = 1 - \exp[-O_\rho(x)] |_{\rho>0, 0<x<1}, \quad (1)$$

where $O_\rho(x) = \left(\frac{x}{1-x}\right)^\rho$. The probability density function (PDF) corresponding to (1) can be expressed as

$$f_\rho(x) = \rho x^{\rho-1} (1-x)^{-\rho-1} \exp[-O_\rho(x)] |_{\rho>0, 0<x<1},$$

where ρ is a shape parameter. More information about the RKi distribution can be explored in Kumar & Dharmaja (2013) and Dey et al. (2019). Al-Babtain et al. (2020) proposed the reduced Kies (RKi-G) family based on the RKi distribution and T-X family (Alzaatreh et al., 2013). The CDF and PDF of the RKi-G family take the forms

$$W_{\rho,\underline{\varphi}}(x) = 1 - \exp \left\{ \left[\frac{G_{\underline{\varphi}}(x)}{1 - G_{\underline{\varphi}}(x)} \right]^\rho \right\}, \quad x > 0, \rho > 0, \quad (2)$$

where $G_{\underline{\varphi}}(x)$ is the baseline CDF depending on a parameter vector $\underline{\varphi}$ and

$$w_{\rho,\underline{\varphi}}(x) = \frac{\rho g_{\underline{\varphi}}(x) G_{\underline{\varphi}}(x)^{\rho-1}}{\left[1 - G_{\underline{\varphi}}(x)\right]^{\rho+1}} \exp \left\{ \left[\frac{G_{\underline{\varphi}}(x)}{1 - G_{\underline{\varphi}}(x)} \right]^\rho \right\}. \quad (3)$$

This paper proposes a flexible extension of the RKi-G family using the exponentiated-G (E-G) family which is the most widely used generalization technique. In fact, this generalization technique can be traced back to Lehmann (1952). This technique received a great deal of attention in the last three decades and more than fifty E-G distributions have already been published. Some notable exponentiated models include the exponentiated-Weibull (Mudholkar & Srivastava, 1993), exponentiated-exponential (Gupta et al., 1998), exponentiated-Gumbel, exponentiated-gamma, and exponentiated-Fréchet (Nadarajah & Kotz, 2006b), exponentiated generalized family (Cordeiro et al., 2013), exponentiated Weibull-Pareto (Afify et al., 2016), exponentiated Burr-Hatke (El-Morshedy et al., 2021), exponentiated new power-function (Al Mutairi et al., 2021), and exponentiated exponential-Weibull (Mastor et al., 2022), among others.

As a result of this, a great attention to practical results and applications to real data have considered. Furthermore, eight classic estimation methods including the maximum Likelihood method, ordinary and weighted least-squares methods, maximum product of spacing method, Cramer-von Mises method, Anderson-Darling and right-tail Anderson-Darling methods and percentile method are presented. All related theoretical derivations, in addition to a comprehensive set of simulation studies, and simulation studies are conducted under certain conditions and with specific controls. The results of the comprehensive simulation are used to assess the estimators behavior of all methods, and a comparison is made between the methods to determine the best and the most efficient one in the estimation processes. To explore the behavior of different estimators of the parameters, numerical simulation results and ranked them with respect to their average of absolute biases, average of mean relative errors and average mean square errors are provided.

In fact, the *exponentiated reduced-Kies-G* (ERKi-G) family is proposed in this paper. Let $G_{\underline{\varphi}}(x)$ be a baseline CDF depending on a parameter vector $\underline{\varphi}$, then the CDF of the ERKi-G family is defined by

$$F_{\rho,\eta,\underline{\varphi}}(x) = \left\{ 1 - \exp \left[-O_{\rho,\underline{\varphi}}(x) \right] \right\}^\eta, \quad x > 0, \rho, \eta > 0, \quad (4)$$

where $O_{\rho,\underline{\varphi}}(x) = \left[\frac{G_{\underline{\varphi}}(x)}{1-G_{\underline{\varphi}}(x)} \right]^{\rho}$. The corresponding PDF of (4) is given by

$$f_{\rho,\eta,\underline{\varphi}}(x) = \frac{\eta \rho g_{\underline{\varphi}}(x) G_{\underline{\varphi}}(x)^{\rho-1}}{\left[1 - G_{\underline{\varphi}}(x) \right]^{\rho+1}} \exp \left[-O_{\rho,\underline{\varphi}}(x) \right] \left\{ 1 - \exp \left[-O_{\rho,\underline{\varphi}}(x) \right] \right\}^{\eta-1}. \quad (5)$$

The hazard rate (HR) function of the ERKi-G family is given from $h_{\rho,\eta,\underline{\varphi}}(x) = f_{\rho,\eta,\underline{\varphi}}(x)/R_{\rho,\eta,\underline{\varphi}}(x)$, where $R_{\rho,\eta,\underline{\varphi}}(x) = 1 - F_{\rho,\eta,\underline{\varphi}}(x)$. Hereafter, a random variable with PDF (5) is denoted by $X \sim \text{ERKi-G}(\rho, \eta, \underline{\varphi})$.

In general, this work is motivated by the following theoretical and practical reasons:

- The right skewed with one peak and symmetric density, the left skewed with one peak and symmetric density, and the right skewed with no peak and extremely heavy tail were all produced by the density of the ERKi-G family (under the exponential model as a special case). This diversity in the form of the density function tells us that the new distribution has wide flexibility, and this flexibility makes the new distribution superior to other competing distributions in statistical and mathematical modeling processes.
- Under the exponential model as a special case, the new failure rate of the new family presented various significant forms, including J-failure rate, monotonically increasing failure rate, monotonically decreasing failure rate and decreasing-constant-increasing failure rate. The new family has a wide range of flexibility, and this flexibility gives the new distribution an advantage over rival distributions in statistical and mathematical modelling procedures. This vast range in the shapes of the hazard rate function informs us of the new family's flexibility.
- In practice, it is possible to choose the new family as the best among all rival distributions, and this decision was well-founded because it was made using eight different comparison criteria. This is true even when all competing statistical distributions can adequately represent the data.
- In the field of data modeling, which is characterized by an increasing failure rate, the new family showed a great advantage in its modeling. This work provides two comprehensive applications to data with an increasing failure rate as an example and proof of this claim.
- Derivation of probability distributions that are flexible enough to keep pace with real data modeling in applied fields such as engineering, actuarial and medical sciences. The skewness coefficient of the ERKi-G family (under the exponential model as a special case) takes positive and negative values and sometimes approaches zero at some parameter values.
- A special model of the ERKi-G family is studied called the ERKi-exponential (ERKiEx) distribution. Two real-life data applications illustrate that the ERKiEx distribution is very flexible competitor to some existing distributions like the gamma and Weibull distributions. It is also a good alternative to some other exponential extensions such as the alpha power Ex Mahdavi & Kundu (2017), generalized odd log-logistic Ex Afify et al. (2020), and extended odd Weibull Ex Afify & Mohamed (2020).

The rest of the paper is organized in six sections. Section 2 deals with the key properties of the proposed family. A special sub-model of the ERKi-G family is introduced in Section 3. In Section 4, the ERKiEx parameters are estimated using different frequentist approaches. The detailed simulation results are also presented in Section 4. Two real-life data sets are fitted in Section 5. The paper is concluded in Section 6.

2 The ERKi-G Properties

In this Section, some important mathematical properties are presented such as the mixture representation of the new PDF and its corresponding CDF, the quantiles, the ordinary moments, incomplete moments and the moment generating function.

2.1 Mixture Representation

Consider the following series

$$\left(1 - \frac{\zeta_1}{\zeta_2}\right)^{\zeta_3} = \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma(1 + \zeta_3)}{j! \Gamma(1 + \zeta_3 - j)} \left(\frac{\zeta_1}{\zeta_2}\right)^j \Big|_{\zeta_3 > 0 \text{ and } |\frac{\zeta_1}{\zeta_2}| < 1}, \quad (6)$$

$$\exp\left(-\frac{\zeta_1}{\zeta_2}\right) = \sum_{k=0}^{\infty} \frac{1}{k!} (-1)^k \left(\frac{\zeta_1}{\zeta_2}\right)^k, \quad (7)$$

and

$$\left(1 - \frac{\zeta_1}{\zeta_2}\right)^{-\zeta_3} = \sum_{h=0}^{\infty} \frac{\Gamma(\zeta_3 + h)}{h! \Gamma(\zeta_3)} \left(\frac{\zeta_1}{\zeta_2}\right)^h \Big|_{\zeta_3 > 0, |\frac{\zeta_1}{\zeta_2}| < 1}. \quad (8)$$

Then, applying (6) to the PDF (5), it follows that

$$f_{\rho, \eta, \underline{\varphi}}(x) = \frac{\eta \rho g_{\underline{\varphi}}(x) G_{\underline{\varphi}}(x)^{\rho-1}}{\left[1 - G_{\underline{\varphi}}(x)\right]^{\rho+1}} \sum_{j=0}^{+\infty} \frac{(-1)^j \Gamma(\eta)}{j! \Gamma(\eta - j)} \exp\left\{-(1+j) O_{\rho, \underline{\varphi}}(x)\right\}.$$

Then, applying (7) to the last equation gives

$$f_{\rho, \eta, \underline{\varphi}}(x) = \eta \rho g_{\underline{\varphi}}(x) \sum_{j,k=0}^{+\infty} \frac{(-1)^{j+k} (1+j)^k \Gamma(\eta)}{j! k! \Gamma(\eta - j)} \frac{G_{\underline{\varphi}}(x)^{\rho k + \rho - 1}}{\left[1 - G_{\underline{\varphi}}(x)\right]^{\rho k + \rho + 1}}.$$

Finally, after expanding the quantity $\left[1 - G_{\underline{\varphi}}(x)\right]^{\rho k + \rho + 1}$ using (8), the RKi-G PDF can be expressed as

$$f_{\rho, \eta, \underline{\varphi}}(x) = \sum_{k,h=0}^{\infty} v_{k,h} g_{\rho k + \rho + h}(x), \quad (9)$$

where $g_{\rho k + \rho + h}(x)$ denotes the exponentiated-G (exp-G) PDF with power parameter $\rho k + \rho + h$, and the coefficient $v_{k,h}$ is given by

$$v_{k,h} = \eta \rho \frac{\Gamma(\eta) \Gamma([\rho k + \rho + 1] + h)}{k! h! \Gamma(\rho k + \rho + 1) (\rho k + \rho + h)} \sum_{j=0}^{\infty} \frac{(-1)^{j+k} (1+j)^k}{j! \Gamma(\eta - j)}.$$

Equation (9) gives the ERKi-G family PDF as a linear combination of exp-G PDFs and enable us to derive some mathematical properties of the ERKi-G family using this representation. Integrating (9), the CDF of X is given by

$$F_{\rho, \eta, \underline{\varphi}}(x) = \sum_{k,h=0}^{\infty} v_{k,h} G_{\rho k + \rho + h}(x), \quad (10)$$

where $G_{\rho k + \rho + h}(x)$ is the CDF of the exp-G family with power parameter $\rho k + \rho + h$. Equations (9) and (10) can easily be employed in deriving many mathematical properties for all the special cases that can result from the use and application of the new family.

2.2 Quantiles and Moments

The quantile function (QF) of (4) is given by

$$x_p = G^{-1} \left(\left\{ 1 + \left[-\log \left(1 - u^{\frac{1}{\eta}} \right) \right]^{-\frac{1}{\rho}} \right\}^{-1}; \underline{\varphi} \right). \quad (11)$$

Based on (11), random samples from (4) can be generated, as $X_i = Q_{ERKi-G}(U_i)$, where $U_i \sim \text{Uniform}(0, 1)$, $i = 1, \dots, n$. Henceforth, $Y_{\rho k + \rho + h}$ denotes a random variable having the exp-G distribution with power parameter $\rho k + \rho + h$. The r^{th} moment of the ERKi-G family follows from (9) as

$$\mu'_{r,X} = E(X^r) = \sum_{k,h=0}^{\infty} v_{k,h} E(Y_{\rho k + \rho + h}^r). \quad (12)$$

The s^{th} incomplete moment of X follows from (9) as

$$\varphi_{s,X}(t) = \sum_{k,h=0}^{\infty} v_{k,h} \int_{-\infty}^t x^s g_{\rho k + \rho + h}(x) dx. \quad (13)$$

The first incomplete moment, $\varphi_{1,X}(t)$, can be used to calculate the mean deviations and the Bonferroni and Lorenz curves (see, Lorenz (1905) and Bonferroni (2013)), and it follows from the last equation with $s = 1$.

2.3 Generating Function

This section provides two expressions for the MGF of X . The first one follows from Equation (9)

$$M_X(t) = \sum_{k,h=0}^{\infty} v_{k,h} M_{\rho k + \rho + h}(t),$$

where $M_{\rho k + \rho + h}(t)$ is the MGF of the random variable $Y_{\rho k + \rho + h}$. Hence, $M_X(t)$ can be determined from the exp-G MGF.

The second formula can also be derived from (9) by considering $u = G(x; \underline{\varphi})$. Therefore, the MGF can be represented as

$$M_X(t) = \sum_{k,h=0}^{\infty} v_{k,h} \tau(t, k), \quad (14)$$

where $\tau(t, k) = \int_0^1 \exp[t Q_G(u)] u^{\rho k + \rho + h - 1} du$ and $Q_G(u)$ is the QF corresponding to $G(x; \underline{\varphi})$, i.e., $Q_G(u) = G^{-1}(u; \underline{\varphi})$. To get the numerical quantifications of (12), (13) and (14), the statistical programs like "R", "MATHCAD" and "MATLAB" will be used.

3 The ERKiEx Distribution

Consider the exponential (Ex) distribution with positive scale parameter λ . Then, the CDF of the Ex model is given by

$$G(z) = 1 - \exp(-\lambda z), \quad z > 0.$$

Then, the r^{th} ordinary and incomplete moments of the Ex model are given, respectively, by $\mu'_r = r!/\lambda^r$ and $\varphi_r(x) = \lambda^{-r} \gamma(1+r, \lambda t)$, where $\gamma(a, b) = \int_0^b y^{a-1} \exp(-y) dy$ is the lower incomplete gamma function.

Now, the CDF of the ERKiEx distribution can be defined by inserting the CDF of the Ex distribution in (4). Hence, the CDF of the ERKiEx distribution takes the form

$$F_{\lambda, \rho, \eta}(x) = [1 - \exp\{-[\exp(\lambda x) - 1]^{\rho}\}]^{\eta}, \quad x > 0, \rho, \eta, \lambda > 0. \quad (15)$$

The corresponding PDF of (15) reduces to

$$f_{\lambda,\rho,\eta}(x) = \eta\rho\lambda [1 - \exp(-\lambda x)]^{\rho-1} \frac{\exp(\rho\lambda x) \exp\{-[\exp(\lambda x) - 1]^\rho\}}{[1 - \exp\{-[\exp(\lambda x) - 1]^\rho\}]^{1-\eta}}, \quad (16)$$

where ρ and η are shape parameters and λ is a scale parameter.

The HR function (HRF) of the ERKiEx distribution reduces to

$$h_{\lambda,\rho,\eta}(x) = \eta\rho\lambda [1 - \exp(-\lambda x)]^{\rho-1} \frac{\exp(\rho\lambda x) \exp\{-[\exp(\lambda x) - 1]^\rho\}}{1 - \exp\{-[\exp(\lambda x) - 1]^\rho\}}. \quad (17)$$

Its reversed HRF has the form

$$r_{\lambda,\rho,\eta}(x) = \eta\rho\lambda [1 - \exp(-\lambda x)]^{\rho-1} \frac{\exp(\rho\lambda x) \exp\{-[\exp(\lambda x) - 1]^\rho\}}{[1 - \exp\{-[\exp(\lambda x) - 1]^\rho\}]}. \quad (18)$$

The QF of the ERKiEx distribution is obtained by inverting (15) as

$$x_p = -\frac{1}{\lambda} \log \left(1 - \left\{ 1 + \left[-\log \left(1 - u^{\frac{1}{\eta}} \right) \right]^{-\frac{1}{\rho}} \right\}^{-1} \right). \quad (19)$$

Note that x_p can be used to generate ERKiEx random variates. Possible shapes of the PDF and HRF the ERKiEx distribution are displayed in Figures 1 and 2 for selected values of λ , ρ and η . The plots shows that the ERKiEx HRF can be decreasing, increasing and bathtub shaped. This is an important feature of the ERKiEx distribution over the Ex distribution, where the last one cannot model phenomenon showing decreasing, increasing and bathtub hazard shapes.

The survival function (SF) of the ERKiEx distribution is given as

$$S(x; \phi) = 1 - [1 - \exp\{-[\exp(\lambda x) - 1]^\rho\}]^\eta.$$

The odds function of the ERKiEx model takes the form

$$O(x; \phi) = \frac{F(x|\phi)}{S(x|\phi)} = \frac{[1 - \exp\{-[\exp(\lambda x) - 1]^\rho\}]^\eta}{1 - [1 - \exp\{-[\exp(\lambda x) - 1]^\rho\}]^\eta}.$$

Proposition *The r -th moment of the ERKiEx distribution is defined by*

$$\begin{aligned} \mu'_r, X &= \sum_{w,u=0}^{\infty} \psi_{w,u} \int_0^{\infty} x^r g_{\vartheta w+u}(x) dx \text{ for } r \in \mathbb{N}. \\ \mu'_{r,X} &= \sum_{k,h=0}^{\infty} v_{k,h} (\rho k + \rho + h)^{\frac{r}{\theta}} \Gamma\left(1 - \frac{r}{\theta}\right). \end{aligned} \quad (19)$$

Setting $r = 1$ in Equation (19), the mean of X is obtained.

The s^{th} incomplete moment of the ERKiEx distribution takes the form

$$\phi_{s,X}(x) = \sum_{k,h=0}^{\infty} v_{k,h} \int_0^t x^s g_{\rho k + \rho + h}(x) dx.$$

Then, ($\forall s < \theta$) has the form

$$\phi_{s,X}(x) = \sum_{k,h=0}^{\infty} v_{k,h} (\rho k + \rho + h)^{\frac{s}{\theta}} \gamma\left(1 - \frac{s}{\theta}, (\rho k + \rho + h)\lambda t^{-\theta}\right). \quad (20)$$

The mean ($\mu(X)$), variance ($\sigma^2(X)$), skewness ($\xi_1(X)$), and kurtosis ($\xi_2(X)$) of the ERKiEx distribution are calculated numerically for different values of λ , ρ and η . Table 1 reports these

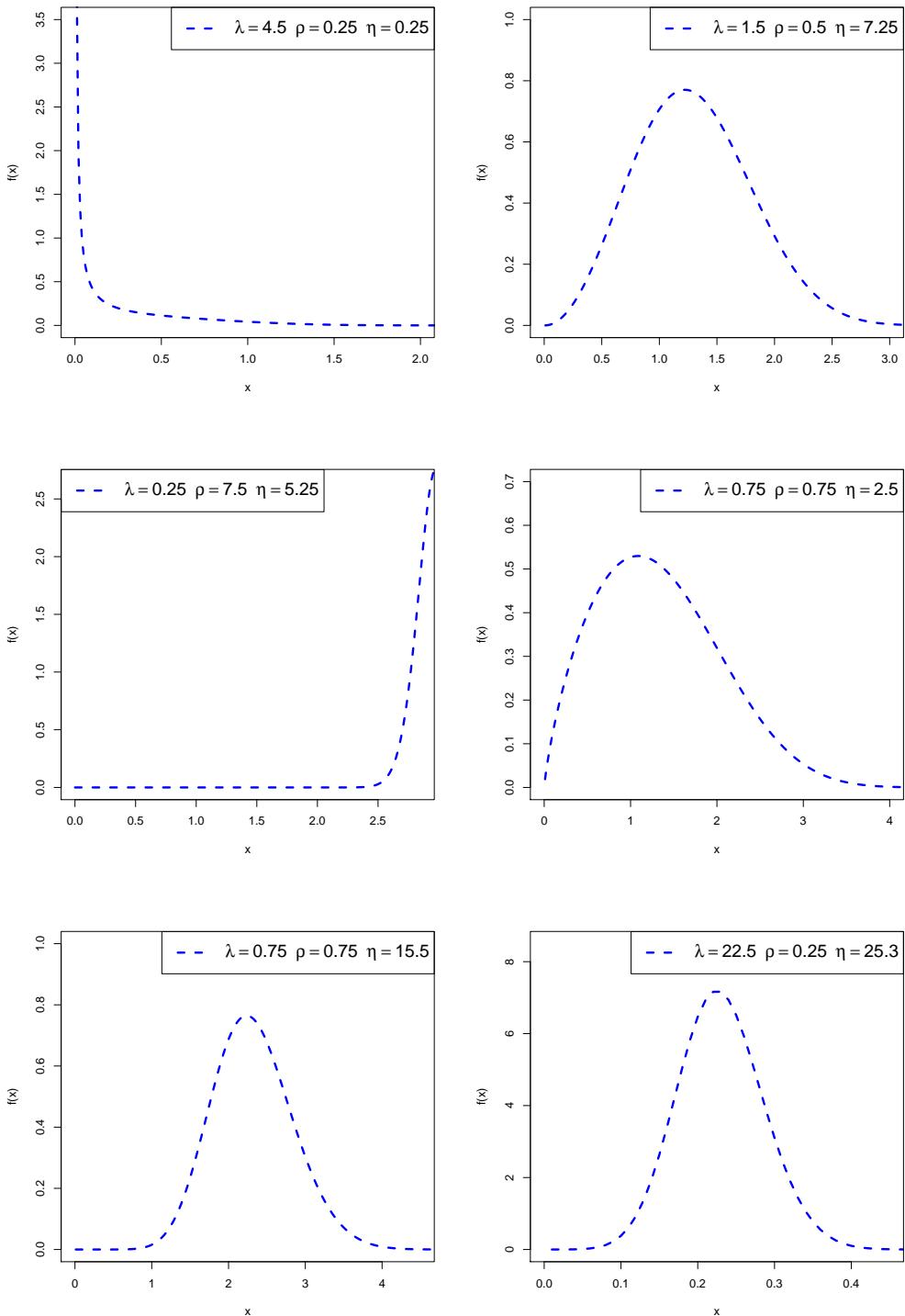


Figure 1: Plots of ERKiEx PDFs for different parametric values

numerical results. Based on Table 1, it is noted that the skewness coefficient of the ERKiEx distribution takes positive and negative values. Also, the kurtosis coefficient takes values greater than 3 and others less than 3, and it sometimes approaches 3 for some parameter values. Diversity in the values of the skewness and kurtosis coefficients is an important indicator for the flexibility of the ERKiEx distribution.

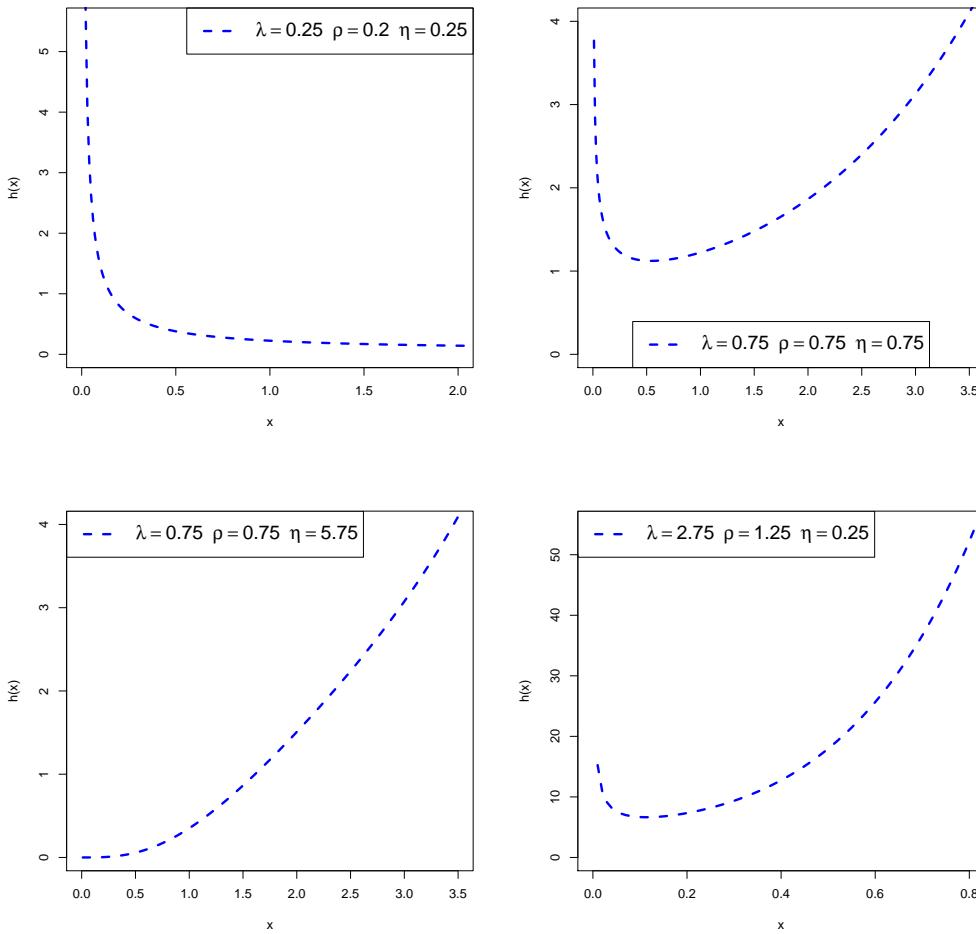


Figure 2: Plots of ERKiEx HRF for different parametric values

Table 1: Some numerical values for $\mu(X)$, $\sigma^2(X)$, $\xi_1(X)$, and $\xi_2(X)$ of the ERKiEx distribution.

ϕ^\top	$\mu(X)$	$\sigma^2(X)$	$\xi_1(X)$	$\xi_2(X)$
($\lambda = 1.50$, $\rho = 0.50$, $\eta = 0.75$)	0.3686	0.2298	1.6849	5.5779
($\lambda = 1.50$, $\rho = 1.25$, $\eta = 0.75$)	0.3406	0.0561	0.6506	2.8224
($\lambda = 1.50$, $\rho = 2.50$, $\eta = 0.75$)	0.3730	0.0216	-0.0627	2.4667
($\lambda = 1.50$, $\rho = 3.50$, $\eta = 0.75$)	0.3909	0.0132	-0.3439	2.7492
($\lambda = 1.50$, $\rho = 0.50$, $\eta = 1.50$)	0.6053	0.2951	1.0277	3.5265
($\lambda = 0.45$, $\rho = 0.50$, $\eta = 0.75$)	0.8029	1.0705	1.3638	3.8078
($\lambda = 0.45$, $\rho = 1.25$, $\eta = 0.75$)	1.1339	0.6199	0.6380	2.7697
($\lambda = 0.45$, $\rho = 2.50$, $\eta = 0.75$)	1.2435	0.2399	-0.0627	2.4667
($\lambda = 0.45$, $\rho = 3.50$, $\eta = 1.25$)	1.4685	0.0969	-0.3685	3.0186
($\lambda = 1.50$, $\rho = 0.50$, $\eta = 2.50$)	0.8196	0.3101	0.6749	2.9232
($\lambda = 3.00$, $\rho = 0.50$, $\eta = 0.75$)	0.1843	0.0575	1.6849	5.5779
($\lambda = 3.00$, $\rho = 1.25$, $\eta = 1.25$)	0.2193	0.0132	0.3905	2.6579
($\lambda = 3.00$, $\rho = 2.50$, $\eta = 2.50$)	0.2539	0.0025	-0.1302	2.9297
($\lambda = 3.00$, $\rho = 3.50$, $\eta = 1.50$)	0.2279	0.0019	-0.3494	3.0610
($\lambda = 3.50$, $\rho = 0.50$, $\eta = 0.75$)	0.1580	0.0422	1.6848	5.5780
($\lambda = 3.50$, $\rho = 1.25$, $\eta = 1.25$)	0.1880	0.0097	0.3905	2.6579
($\lambda = 3.50$, $\rho = 2.50$, $\eta = 2.50$)	0.2177	0.0018	-0.1302	2.9297

4 Estimation and Simulations

In this section, the ERKiEx parameters λ , ρ , and η are estimated using different frequentist approaches. These methods can not be compared analytically, hence detailed simulation results are provided to address their performances and compare between them using partial and overall ranks.

4.1 Estimation Methods

Let x_1, \dots, x_n be a sample from the ERKiEx distribution in (16). Hence, the log likelihood function of the ERKiEx distribution, for $\phi = (\lambda, \rho, \eta)^\top$, takes the form

$$\begin{aligned} l(\phi; \mathbf{x}) &= n \log(\eta) + n \log(\rho) + n \log(\lambda) + \rho \lambda \sum_{i=1}^n x_i \\ &\quad + (\rho - 1) \sum_{i=1}^n \log \{1 - \exp(-\lambda x_i)\} - \sum_{i=1}^n \exp\{-[\exp(\lambda x_i) - 1]^\rho\} \\ &\quad + (\eta - 1) \sum_{i=1}^n \log(1 - \exp\{-[\exp(\lambda x_i) - 1]^\rho\}) \end{aligned} \quad (21)$$

From the expressions $\frac{\partial}{\partial \lambda} l(\phi; \mathbf{x}) = 0$, $\frac{\partial}{\partial \rho} l(\phi; \mathbf{x}) = 0$, and $\frac{\partial}{\partial \eta} l(\phi; \mathbf{x}) = 0$, the likelihood equations are

$$\begin{aligned} \frac{\partial}{\partial \lambda} l(\phi; \mathbf{x}) &= \frac{n}{\lambda} + \rho \sum_{i=1}^n x_i + (\rho - 1) \lambda \sum_{i=1}^n \frac{x_i e^{-2\lambda x_i}}{w_i} - \rho \sum_{i=1}^n x_i e^{\lambda x_i} w_i^{\rho-1} + (\eta - 1) \rho \sum_{i=1}^n \frac{x_i e^{\lambda x_i} w_i^{\rho-1}}{1 - e^{-w_i^\rho}} = 0, \\ \frac{\partial}{\partial \rho} l(\phi; \mathbf{x}) &= \frac{n}{\rho} + \lambda \sum_{i=1}^n x_i + \sum_{i=1}^n e^{-\lambda x_i} w_i - \sum_{i=1}^n w_i^\rho \log w_i + (\eta - 1) \sum_{i=1}^n \frac{w_i^\rho \log w_i}{\exp(w_i^\rho) - 1} = 0 \end{aligned}$$

and

$$\frac{\partial}{\partial \eta} l(\phi; \mathbf{x}) = \frac{n}{\eta} + \sum_{i=1}^n (1 - \exp(-w_i^\rho)) = 0,$$

where $w_i = (e^{\lambda x_i} - 1)$.

The maximum likelihood estimators (MLEs) are obtained by solving the above three equations by using several programs such as SAS (PROC NLMIXED) or R (optim function).

Let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be the order statistics of a random sample from the PDF (16), then the least-squares estimators (LSEs) of the ERKiEx parameters are obtained by solving the following equations

$$\sum_{i=1}^n \left\{ F_{\lambda, \rho, \eta}(x_{(i)}) - \frac{i}{n+1} \right\} \Omega_k(x_{(i)} | \boldsymbol{\eta}) = 0, \quad k = 1, 2, 3,$$

where

$$\Omega_1(x_{(i)} | \boldsymbol{\phi}) = \frac{\partial}{\partial \lambda} F_{\lambda, \rho, \eta}(x_{(i)} | \boldsymbol{\phi}) = \rho \eta x_i e^{\lambda x_i} w_i^\rho \exp(-w_i^\rho) [1 - \exp(-w_i^\rho)]^{\eta-1}, \quad (22)$$

$$\Omega_2(x_{(i)} | \boldsymbol{\phi}) = \frac{\partial}{\partial \rho} F_{\lambda, \rho, \eta}(x_{(i)} | \boldsymbol{\phi}) = \eta \exp(-w_i^\rho) w_i^\rho \log w_i [1 - \exp(-w_i^\rho)]^{\eta-1} \quad (23)$$

and

$$\Omega_3(x_{(i)} | \boldsymbol{\phi}) = \frac{\partial}{\partial \eta} F_{\lambda, \rho, \eta}(x_{(i)} | \boldsymbol{\phi}) = [1 - \exp(-w_i^\rho)]^\eta \log [1 - \exp(-w_i^\rho)]. \quad (24)$$

The solution of Ω_k for $k = 1, 2, 3$ may be obtained numerically. The weighted-LSEs (WLSEs) of the ERKiEx parameters can be determined by solving the following equations:

$$\sum_{i=1}^n \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F_{\lambda,\rho,\eta}(x_{(i)}|\phi) - \frac{i}{n+1} \right] \Omega_k(x_{(i)}|\eta) = 0|_{k=1,2,3},$$

where Ω_k are provided in (22)-(24).

The maximum product of spacings (MPS) is a useful alternative to the maximum likelihood method.

The uniform spacings of a random sample from the ERKiEx distribution are defined (for $i = 1, 2, \dots, n+1$) by $D_i(\phi) = F_{\lambda,\rho,\eta}(x_{(i)}|\phi) - F_{\lambda,\rho,\eta}(x_{(i-1)}|\phi)$, where $F_{\lambda,\rho,\eta}(x_{(0)}|\phi) = 0$, $F_{\lambda,\rho,\eta}(x_{(n+1)}|\phi) = 1$ and $\sum_{i=1}^{n+1} D_i(\phi) = 1$.

The MPS estimators (MPSEs) of the ERKiEx parameters are determined by maximizing the following geometric mean (GM) of spacings

$$G(\phi) = \left[\prod_{i=1}^{n+1} D_i(\phi) \right]^{\frac{1}{n+1}},$$

with respect to the ERKiEx parameters. The MPSEs of the ERKiEx parameters are obtained by solving the following equations

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\phi)} [\Omega_k(x_{(i)}|\phi) - \Omega_k(x_{(i-1)}|\phi)] = 0,$$

where Ω_k are provided in (22)-(24) for $k = 1, 2, 3$.

The Cramér–von Mises estimators (CVMEs) are obtained as the difference between the estimated and empirical CDFs. The CRVMEs of the ERKiEx parameters are determined by solving the following equations:

$$\sum_{i=1}^n \left[F_{\lambda,\rho,\eta}(x_{(i)}|\phi) - \frac{2i-1}{2n} \right] \Omega_k(x_{(i)}|\phi) = 0,$$

where Ω_k are provided in (22)-(24) for $k = 1, 2, 3$.

The Anderson–Darling estimators (ADEs) of the ERKiEx parameters are calculated by minimizing

$$A(\phi) = -n - \frac{1}{n} \sum_{i=1}^n (2i-1) [\log F_{\lambda,\rho,\eta}(x_{(i)}|\phi) + \log S_{\lambda,\rho,\eta}(x_{(i)}|\phi)],$$

with respect to λ, ρ , and η .

The right-tail Anderson–Darling estimators (RADEs) of the ERKiEx parameters are calculated by solving the following equations

$$-2 \sum_{i=1}^n \Omega_k(x_{i:n}|\phi) + \frac{1}{n} \sum_{i=1}^n (2i-1) \frac{\Omega_k(x_{n+1-i:n}|\phi)}{S_{\lambda,\rho,\eta}(x_{n+1-i:n}|\phi)} = 0,$$

where Ω_k are provided in (22)-(24) for $k = 1, 2, 3$.

Consider the unbiased estimator of $F_{\lambda,\rho,\eta}(x_{(i)}|\phi)$ which is defined by $u_i = i/(n+1)$. The percentile-based estimators (PCEs) of the ERKiEx parameters are calculated by minimizing

$$P(\phi) = \sum_{i=1}^n \left(x_{(i)} - \frac{1}{\lambda} \log \left\{ 1 + \left(-\log \left[1 - p^{\frac{1}{\rho}} \right] \right)^{\frac{1}{\rho}} \right\} \right)^2,$$

with respect to λ, ρ , and η .

4.2 Simulation Analysis

The behavior of different introduced estimators of the ERKiEx parameters is explored by using numerical simulations. Additionally, the average of mean relative errors (MREs), average of absolute biases ($|BIAS|$), and average mean square errors (MSEs) are calculated and ranked to order the estimates accordingly. These measures are specified by the following equations:

- $MREs = \frac{1}{N} \sum_{i=1}^N |\hat{\phi} - \phi|/\phi,$
- $(|BIAS(\hat{\phi})|), |Bias(\hat{\phi})| = \frac{1}{N} \sum_{i=1}^N |\hat{\phi} - \phi| \text{ and}$
- $MSEs = \frac{1}{N} \sum_{i=1}^N (\hat{\phi} - \phi)^2.$

These three measures are calculated and then ranked for all estimates and all sample sizes through generating $N = 5000$ random samples x_1, x_2, \dots, x_N of sizes $n = 20, 50, 80, 150$, and 500 from the ERKiEx distribution using its QF (11). The following parametric values $\lambda = \{0.5, 0.45, 3\}$, $\rho = \{0.35, 1.25, 2.5, 3.5\}$, and $\eta = \{0.75, 1.25, 2.5\}$, are considered and all results are obtained using **R** software (version 4.0.2) R Core Team (2020). For each sample and each parametric combination, the ERKiEx parameters λ , ρ , and η are estimated using the eight estimators: MLEs, LSEs, WLSEs, MPSEs, CVMEs, ADEs, PCEs, and RADEs.

Four out of twelve simulated results are listed in Tables (2–5). These tables also indicate the ranks of each of the proposed estimates in each row, where the superscripts refer to the indicators, and $\sum Ranks$ refers to the partial sum of ranks in each column for a particular n .

The partial and overall ranks for all parametric combinations are reported in Table 6. In conclusion, based on Tables 2–5, all methods of estimation illustrate the consistency property for all parametric combinations. Table 6 shows that the MPS approach has an overall score of 115.5, hence it outperforms other estimation methods. Therefore, the MPS method shows the superiority in estimating the ERKiEx parameters.

5 Data Analysis

In this section, the applicability of the ERKiEx distribution is checked by using two real-life data sets from applied fields. The first data represents failure times of a particular windshield device and it contains 74 observations Murthy et al. (2004). The second data contains 40 observations about time-to-failure (10^{3h}) of turbocharger of one type of engine Xu et al. (2003). This data is analyzed by Cordeiro et al. (2019).

Furthermore, the fitting performance of the ERKiEx distribution is compared with some other competitive models based on goodness-of-fit measures including the maximized log-likelihood ($-2\hat{\ell}$), Akaike information (AIC), consistent Akaike information (CAIC), Hannan–Quinn information (HQIC), and Bayesian information (BIC). Additionally, some other statistics are considered for comparison such as Anderson–Darling (A^*), Cramér–von Mises (W^*), Kolmogorov–Smirnov (KS) statistics with its p-value (PV).

The new ERKiEx model is compared with some rival models including the reduced-Kies exponential (RKiEx), Marshall–Olkin exponential (MOEx) Marshall & Olkin (1997), Kumaraswamy exponential (KwEx), gamma (Ga), exponentiated exponential (EEx) Gupta & Kundu (2001), and Ex distributions.

Table 2: Simulation results for $\phi = (\lambda = 0.5, \rho = 0.35, \eta = 0.75)^\top$.

<i>n</i>	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRMES	MPSES	PCEs	ADEs	RADES
20	MSE	$\hat{\lambda}$	0.46707 {4}	0.56214 {6}	0.51058 {5}	0.59180 {7}	0.45622 {3}	1.32447 {8}	0.42569 {1}	0.44204 {2}
		$\hat{\rho}$	0.17087 {1}	0.22335 {6}	0.22002 {5}	0.21109 {4}	0.22993 {7}	0.35893 {8}	0.20142 {3}	0.19316 {2}
		$\hat{\eta}$	0.43346 {3}	0.45042 {6}	0.45044 {7}	0.44446 {5}	0.43987 {4}	0.46558 {8}	0.41854 {1}	0.43046 {2}
		$\hat{\lambda}$	0.55286 {4}	1.00984 {6}	0.79748 {5}	1.10796 {7}	0.50599 {1}	9.29754 {8}	0.50992 {2}	0.51295 {3}
	MRE	$\hat{\rho}$	0.07018 {1}	0.12647 {7}	0.10925 {5}	0.11814 {6}	0.10783 {4}	0.67998 {8}	0.08499 {2}	0.08503 {3}
		$\hat{\eta}$	0.33042 {5}	0.34384 {7}	0.34023 {6}	0.35275 {8}	0.31083 {3}	0.31868 {4}	0.28832 {1}	0.30344 {2}
		$\hat{\lambda}$	0.93414 {4}	1.12429 {6}	1.02116 {5}	1.18360 {7}	0.91244 {3}	2.64894 {8}	0.85137 {1}	0.88408 {2}
		$\hat{\rho}$	0.48820 {1}	0.63814 {6}	0.62864 {5}	0.60311 {4}	0.65694 {7}	1.02550 {8}	0.57548 {3}	0.55190 {2}
	$\sum Ranks$		26 {3}	56 {7}	50 {5}	53 {6}	36 {4}	68 {8}	15 {1}	20 {2}
50	MSE	$\hat{\lambda}$	0.27465 {3}	0.33427 {7}	0.30121 {5}	0.33108 {6}	0.27977 {4}	0.51111 {8}	0.26894 {2}	0.26642 {1}
		$\hat{\rho}$	0.14022 {1}	0.16455 {5}	0.16394 {4}	0.15910 {3}	0.19604 {7}	0.23029 {8}	0.15507 {2}	0.16482 {6}
		$\hat{\eta}$	0.30217 {1}	0.34154 {7}	0.33258 {6}	0.33042 {5}	0.32946 {4}	0.36079 {8}	0.31348 {2}	0.31617 {3}
		$\hat{\lambda}$	0.14668 {4}	0.26478 {6}	0.19339 {5}	0.26745 {7}	0.13889 {2}	0.80867 {8}	0.14655 {3}	0.13488 {1}
	MRE	$\hat{\rho}$	0.04160 {1}	0.05496 {6}	0.05120 {4}	0.05178 {5}	0.07837 {7}	0.07844 {8}	0.04557 {2}	0.04760 {3}
		$\hat{\eta}$	0.14630 {2}	0.17520 {8}	0.16441 {5}	0.17078 {6}	0.15680 {4}	0.17516 {7}	0.14823 {3}	0.14367 {1}
		$\hat{\lambda}$	0.54929 {3}	0.66854 {7}	0.60241 {5}	0.66216 {6}	0.55954 {4}	1.02222 {8}	0.53788 {2}	0.53285 {1}
		$\hat{\rho}$	0.40063 {1}	0.47014 {5}	0.46841 {4}	0.45456 {3}	0.56013 {7}	0.65796 {8}	0.44305 {2}	0.47092 {6}
	$\sum Ranks$		17 {1}	58 {7}	44 {5}	46 {6}	43 {4}	71 {8}	20 {2}	25 {3}
80	MSE	$\hat{\lambda}$	0.22125 {3}	0.26849 {7}	0.23237 {5}	0.26259 {6}	0.22076 {2}	0.36991 {8}	0.22406 {4}	0.21686 {1}
		$\hat{\rho}$	0.12432 {1}	0.15422 {6}	0.14009 {2}	0.14696 {5}	0.17781 {7}	0.21851 {8}	0.14330 {3}	0.14574 {4}
		$\hat{\eta}$	0.25572 {1}	0.29946 {7}	0.27714 {4}	0.29001 {6}	0.27987 {5}	0.32726 {8}	0.27659 {3}	0.27289 {2}
		$\hat{\lambda}$	0.08716 {4}	0.14785 {7}	0.10125 {5}	0.14252 {6}	0.07756 {2}	0.30564 {8}	0.08635 {3}	0.07683 {1}
	MRE	$\hat{\rho}$	0.03347 {1}	0.04781 {6}	0.03991 {3}	0.04382 {5}	0.06722 {7}	0.06903 {8}	0.04051 {4}	0.03989 {2}
		$\hat{\eta}$	0.10497 {2}	0.12782 {7}	0.11241 {4}	0.12381 {6}	0.11407 {5}	0.14123 {8}	0.10941 {3}	0.10346 {1}
		$\hat{\lambda}$	0.44251 {3}	0.53698 {7}	0.46473 {5}	0.52518 {6}	0.44151 {2}	0.73982 {8}	0.44811 {4}	0.43371 {1}
		$\hat{\rho}$	0.35521 {1}	0.44064 {6}	0.40025 {2}	0.41987 {5}	0.50803 {7}	0.62433 {8}	0.40943 {3}	0.41641 {4}
	$\sum Ranks$		17 {1}	60 {7}	34 {4}	51 {6}	42 {5}	72 {8}	30 {3}	18 {2}
150	MSE	$\hat{\lambda}$	0.17718 {4}	0.20887 {7}	0.18125 {5}	0.20584 {6}	0.17261 {2}	0.26180 {8}	0.17307 {3}	0.17068 {1}
		$\hat{\rho}$	0.10857 {1}	0.12698 {6}	0.11724 {2}	0.12324 {4}	0.14853 {7}	0.19458 {8}	0.11747 {3}	0.12372 {5}
		$\hat{\eta}$	0.21184 {1}	0.24575 {7}	0.22278 {4}	0.24224 {6}	0.22688 {5}	0.27440 {8}	0.21906 {2}	0.22021 {3}
		$\hat{\lambda}$	0.05647 {5}	0.07373 {6}	0.05513 {4}	0.07492 {7}	0.04567 {2}	0.11904 {8}	0.04791 {3}	0.04396 {1}
	MRE	$\hat{\rho}$	0.02680 {1}	0.03411 {6}	0.03026 {2}	0.03178 {4}	0.05106 {7}	0.06077 {8}	0.03048 {3}	0.03228 {5}
		$\hat{\eta}$	0.07606 {4}	0.08618 {7}	0.07375 {3}	0.08362 {6}	0.07849 {5}	0.10219 {8}	0.07081 {2}	0.06996 {1}
		$\hat{\lambda}$	0.35435 {4}	0.41775 {7}	0.36251 {5}	0.41167 {6}	0.34521 {2}	0.52361 {8}	0.34614 {3}	0.34136 {1}
		$\hat{\rho}$	0.31021 {1}	0.36280 {6}	0.33498 {2}	0.35211 {4}	0.42437 {7}	0.55596 {8}	0.33564 {3}	0.35349 {5}
	$\sum Ranks$		22 {1}	59 {7}	31 {4}	49 {6}	42 {5}	72 {8}	24 {2}	25 {3}
500	MSE	$\hat{\lambda}$	0.11852 {5}	0.13041 {7}	0.10927 {4}	0.12970 {6}	0.09240 {1}	0.15722 {8}	0.10608 {3}	0.10600 {2}
		$\hat{\rho}$	0.07540 {5}	0.08409 {7}	0.07428 {2}	0.08136 {6}	0.07188 {1}	0.13318 {8}	0.07476 {3}	0.07535 {4}
		$\hat{\eta}$	0.14056 {4}	0.15978 {7}	0.14062 {5}	0.15866 {6}	0.12225 {1}	0.18286 {8}	0.13752 {2}	0.13876 {3}
		$\hat{\lambda}$	0.03447 {7}	0.02631 {6}	0.01928 {4}	0.02586 {5}	0.01554 {1}	0.03793 {8}	0.01850 {3}	0.01783 {2}
	MRE	$\hat{\rho}$	0.01705 {6}	0.01747 {7}	0.01485 {1}	0.01564 {4}	0.01620 {5}	0.03806 {8}	0.01560 {3}	0.01523 {2}
		$\hat{\eta}$	0.04609 {7}	0.03944 {6}	0.03223 {4}	0.03812 {5}	0.02886 {1}	0.05397 {8}	0.03194 {3}	0.03179 {2}
		$\hat{\lambda}$	0.23703 {5}	0.26082 {7}	0.21853 {4}	0.25939 {6}	0.18481 {1}	0.31444 {8}	0.21216 {3}	0.21200 {2}
		$\hat{\rho}$	0.21543 {5}	0.24025 {7}	0.21222 {2}	0.23245 {6}	0.20538 {1}	0.38052 {8}	0.21360 {3}	0.21529 {4}
	$\sum Ranks$		48 {5}	61 {7}	31 {4}	50 {6}	13 {1}	72 {8}	25 {3}	24 {2}

Table 3: Simulation results for $\phi = (\lambda = 0.5, \rho = 1.25, \eta = 0.75)^\top$.

<i>n</i>	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\lambda}$	0.26082 {8}	0.21559 {3}	0.21723 {4}	0.24105 {7}	0.18606 {1}	0.19494 {2}	0.22398 {5}	0.22892 {6}
		$\hat{\rho}$	1.73213 {6}	1.69802 {5}	1.45416 {3}	1.82868 {8}	1.42084 {2}	1.65212 {4}	1.31386 {1}	1.74025 {7}
		$\hat{\eta}$	0.77273 {8}	0.61917 {4}	0.61546 {3}	0.69730 {7}	0.54558 {1}	0.54904 {2}	0.62175 {5}	0.66113 {6}
		$\hat{\lambda}$	0.10947 {7}	0.07606 {4}	0.07284 {2}	0.09331 {6}	0.05222 {1}	0.07383 {3}	0.07617 {5}	0.11036 {8}
	MSE	$\hat{\rho}$	6.60392 {4}	6.78180 {5}	5.18590 {2}	7.99293 {7}	5.27229 {3}	7.06083 {6}	4.27688 {1}	8.75368 {8}
		$\hat{\eta}$	0.90877 {8}	0.61010 {5}	0.58043 {3}	0.74891 {6}	0.43814 {1}	0.43900 {2}	0.58687 {4}	0.76260 {7}
		$\hat{\lambda}$	0.52164 {8}	0.43117 {3}	0.43446 {4}	0.48210 {7}	0.37212 {1}	0.38988 {2}	0.44797 {5}	0.45785 {6}
		$\hat{\rho}$	1.38570 {6}	1.35841 {5}	1.16333 {3}	1.46295 {8}	1.13667 {2}	1.32169 {4}	1.05109 {1}	1.39220 {7}
	MRE	$\hat{\eta}$	1.03030 {8}	0.82556 {4}	0.82061 {3}	0.92973 {7}	0.72744 {1}	0.73206 {2}	0.82899 {5}	0.88151 {6}
		$\sum Ranks$	63 {7.5}	38 {5}	27 {2.5}	63 {7.5}	13 {1}	27 {2.5}	32 {4}	61 {6}
50	BIAS	$\hat{\lambda}$	0.22059 {8}	0.18914 {4}	0.18571 {3}	0.19884 {7}	0.16012 {1}	0.17635 {2}	0.19287 {5}	0.19554 {6}
		$\hat{\rho}$	0.92304 {6}	1.12016 {8}	0.89439 {4}	1.10941 {7}	0.74467 {1}	0.86559 {3}	0.76478 {2}	0.89748 {5}
		$\hat{\eta}$	0.60897 {8}	0.54057 {5}	0.52270 {3}	0.56334 {7}	0.44843 {1}	0.49115 {2}	0.52685 {4}	0.55271 {6}
		$\hat{\lambda}$	0.08583 {8}	0.05158 {3}	0.05282 {4}	0.05956 {7}	0.04144 {1}	0.04268 {2}	0.05778 {5}	0.05943 {6}
	MSE	$\hat{\rho}$	1.77349 {3}	2.94241 {8}	1.81845 {4}	2.92982 {7}	1.34533 {2}	1.92690 {5}	1.20929 {1}	2.13733 {6}
		$\hat{\eta}$	0.60247 {8}	0.39894 {4}	0.39440 {3}	0.44274 {6}	0.31838 {1}	0.32650 {2}	0.40774 {5}	0.44363 {7}
		$\hat{\lambda}$	0.44119 {8}	0.37827 {4}	0.37142 {3}	0.39768 {7}	0.32024 {1}	0.35270 {2}	0.38573 {5}	0.39108 {6}
		$\hat{\rho}$	0.73843 {6}	0.89613 {8}	0.71551 {4}	0.88753 {7}	0.59574 {1}	0.69247 {3}	0.61182 {2}	0.71798 {5}
	$\sum Ranks$	$\hat{\eta}$	0.81195 {8}	0.72075 {5}	0.69693 {3}	0.75112 {7}	0.59791 {1}	0.65487 {2}	0.70246 {4}	0.73695 {6}
		$\sum Ranks$	63 {8}	49 {5}	31 {3}	62 {7}	10 {1}	23 {2}	33 {4}	53 {6}
80	BIAS	$\hat{\lambda}$	0.19843 {8}	0.17606 {5}	0.17660 {6}	0.18533 {7}	0.14474 {1}	0.16616 {2}	0.17051 {3}	0.17320 {4}
		$\hat{\rho}$	0.68443 {6}	0.82146 {7}	0.66921 {4}	0.84323 {8}	0.55693 {1}	0.65340 {3}	0.62347 {2}	0.68137 {5}
		$\hat{\eta}$	0.54135 {8}	0.49420 {6}	0.48767 {5}	0.52436 {7}	0.40214 {1}	0.46486 {2}	0.46710 {3}	0.48599 {4}
		$\hat{\lambda}$	0.07345 {8}	0.04682 {4}	0.05057 {6}	0.05297 {7}	0.03796 {1}	0.03870 {2}	0.04842 {5}	0.04463 {3}
	MSE	$\hat{\rho}$	0.90149 {3}	1.46800 {7}	0.93301 {5}	1.58204 {8}	0.68019 {1}	0.92817 {4}	0.74644 {2}	1.04596 {6}
		$\hat{\eta}$	0.50032 {8}	0.34204 {5}	0.35829 {6}	0.39113 {7}	0.27899 {1}	0.29973 {2}	0.33767 {3}	0.34172 {4}
		$\hat{\lambda}$	0.39685 {8}	0.35212 {5}	0.35320 {6}	0.37065 {7}	0.28947 {1}	0.33232 {2}	0.34103 {3}	0.34639 {4}
		$\hat{\rho}$	0.54754 {6}	0.65716 {7}	0.53537 {4}	0.67458 {8}	0.44555 {1}	0.52272 {3}	0.49878 {2}	0.54509 {5}
	$\sum Ranks$	$\hat{\eta}$	0.72180 {8}	0.65894 {6}	0.65022 {5}	0.69914 {7}	0.53618 {1}	0.61982 {2}	0.62280 {3}	0.64799 {4}
		$\sum Ranks$	63 {7}	52 {6}	47 {5}	66 {8}	9 {1}	22 {2}	26 {3}	39 {4}
150	BIAS	$\hat{\lambda}$	0.16346 {8}	0.15423 {6}	0.14916 {2}	0.16186 {7}	0.12441 {1}	0.15051 {4}	0.15214 {5}	0.15015 {3}
		$\hat{\rho}$	0.48142 {4}	0.58352 {7}	0.49479 {5}	0.60749 {8}	0.38418 {1}	0.46792 {2}	0.46816 {3}	0.49590 {6}
		$\hat{\eta}$	0.43794 {7}	0.43429 {6}	0.40951 {2}	0.45748 {8}	0.34034 {1}	0.42196 {5}	0.41462 {3}	0.41990 {4}
		$\hat{\lambda}$	0.05820 {8}	0.03862 {4}	0.03969 {5}	0.04289 {7}	0.03183 {1}	0.03465 {2}	0.04206 {6}	0.03586 {3}
	MSE	$\hat{\rho}$	0.39968 {4}	0.67195 {7}	0.42696 {6}	0.70279 {8}	0.27052 {1}	0.35676 {2}	0.35884 {3}	0.41796 {5}
		$\hat{\eta}$	0.37987 {8}	0.28409 {5}	0.27670 {4}	0.31562 {7}	0.22391 {1}	0.26334 {2}	0.28927 {6}	0.26824 {3}
		$\hat{\lambda}$	0.32692 {8}	0.30847 {6}	0.29832 {2}	0.32371 {7}	0.24882 {1}	0.30102 {4}	0.30429 {5}	0.30031 {3}
		$\hat{\rho}$	0.38513 {4}	0.46681 {7}	0.39583 {5}	0.48599 {8}	0.30735 {1}	0.37434 {2}	0.37453 {3}	0.39672 {6}
	$\sum Ranks$	$\hat{\eta}$	0.58391 {7}	0.57906 {6}	0.54602 {2}	0.60997 {8}	0.45379 {1}	0.56262 {5}	0.55283 {3}	0.55987 {4}
		$\sum Ranks$	58 {7}	54 {6}	33 {3}	68 {8}	9 {1}	28 {2}	37 {4.5}	37 {4.5}
500	BIAS	$\hat{\lambda}$	0.09195 {4}	0.10759 {8}	0.09072 {2}	0.10703 {7}	0.06837 {1}	0.10120 {6}	0.09188 {3}	0.09766 {5}
		$\hat{\rho}$	0.24822 {2}	0.34293 {8}	0.27711 {4}	0.34231 {7}	0.19782 {1}	0.28483 {5}	0.27401 {3}	0.29108 {6}
		$\hat{\eta}$	0.24821 {2}	0.30801 {8}	0.25488 {3}	0.30538 {7}	0.18666 {1}	0.28796 {6}	0.25600 {4}	0.27909 {5}
		$\hat{\lambda}$	0.02571 {8}	0.02306 {6}	0.01887 {3}	0.02322 {7}	0.01297 {1}	0.01939 {4.5}	0.01939 {4.5}	0.01840 {2}
	MSE	$\hat{\rho}$	0.10315 {2}	0.18472 {7}	0.12104 {4}	0.19039 {8}	0.07778 {1}	0.12134 {5}	0.11762 {3}	0.13172 {6}
		$\hat{\eta}$	0.16588 {6}	0.17443 {8}	0.13615 {2}	0.17364 {7}	0.09187 {1}	0.14973 {5}	0.13738 {3}	0.14252 {4}
		$\hat{\lambda}$	0.18390 {4}	0.21518 {8}	0.18144 {2}	0.21405 {7}	0.13675 {1}	0.20239 {6}	0.18377 {3}	0.19532 {5}
		$\hat{\rho}$	0.19857 {2}	0.27434 {8}	0.22168 {4}	0.27385 {7}	0.15825 {1}	0.22787 {5}	0.21921 {3}	0.23286 {6}
	$\sum Ranks$	$\hat{\eta}$	0.33095 {2}	0.41068 {8}	0.33984 {3}	0.40717 {7}	0.24888 {1}	0.38395 {6}	0.34133 {4}	0.37212 {5}
		$\sum Ranks$	32 {4}	69 {8}	27 {2}	64 {7}	9 {1}	49 {6}	30.5 {3}	44 {5}

Table 4: Simulation results for $\phi = (\lambda = 0.5, \rho = 2.25, \eta = 0.75)^T$.

<i>n</i>	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRMVEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\lambda}$	0.26278 {6}	0.24575 {4}	0.25750 {5}	0.28196 {7}	0.20643 {1}	0.21830 {2}	0.23241 {3}	0.28704 {8}
		$\hat{\rho}$	2.86779 {4}	3.16743 {6}	2.88325 {5}	3.43419 {8}	2.62260 {3}	2.54591 {2}	2.42487 {1}	3.21627 {7}
		$\hat{\eta}$	1.41413 {3}	1.58569 {5}	1.58604 {6}	1.94916 {8}	1.17813 {1}	1.22289 {2}	1.45107 {4}	1.94777 {7}
	MSE	$\hat{\lambda}$	0.14853 {1}	0.15464 {2}	0.21275 {7}	0.20261 {6}	0.17456 {4}	0.17511 {5}	0.17041 {3}	0.24448 {8}
		$\hat{\rho}$	20.61780 {5}	23.42130 {6}	19.29146 {4}	26.98077 {7}	17.42400 {3}	13.96011 {2}	13.84724 {1}	28.48377 {8}
		$\hat{\eta}$	6.79032 {3}	9.29840 {6}	8.39895 {5}	12.45742 {8}	4.09189 {1}	4.36121 {2}	6.97246 {4}	12.35703 {7}
50	MRE	$\hat{\lambda}$	0.52556 {6}	0.49151 {4}	0.51501 {5}	0.56392 {7}	0.41286 {1}	0.43659 {2}	0.46483 {3}	0.57409 {8}
		$\hat{\rho}$	1.14711 {4}	1.26697 {6}	1.15330 {5}	1.37368 {8}	1.04904 {3}	1.01837 {2}	0.96995 {1}	1.28651 {7}
		$\hat{\eta}$	1.88550 {3}	2.11425 {5}	2.11472 {6}	2.59889 {8}	1.57084 {1}	1.63053 {2}	1.93476 {4}	2.59703 {7}
	BIAS	$\sum Ranks$	35 {4}	44 {5}	48 {6}	67 {7.5}	18 {1}	21 {2}	24 {3}	67 {7.5}
		$\hat{\lambda}$	0.18225 {7}	0.18078 {6}	0.15160 {4}	0.18059 {5}	0.11505 {1}	0.14210 {3}	0.13196 {2}	0.18782 {8}
		$\hat{\rho}$	1.36746 {4}	1.95537 {7}	1.54773 {5}	1.99848 {8}	1.25772 {1}	1.29395 {2}	1.33381 {3}	1.55944 {6}
80	MSE	$\hat{\eta}$	0.75051 {2}	1.10566 {6}	0.89400 {5}	1.11942 {7}	0.66152 {1}	0.76957 {3}	0.78769 {4}	1.17458 {8}
		$\hat{\lambda}$	0.08567 {4}	0.09595 {6}	0.07675 {3}	0.09573 {5}	0.04033 {1}	0.10772 {7}	0.05343 {2}	0.11792 {8}
		$\hat{\rho}$	4.42310 {4}	8.78270 {7}	5.39699 {5}	9.22178 {8}	3.96310 {3}	3.51851 {1}	3.76460 {2}	6.08845 {6}
	MRE	$\hat{\eta}$	2.06332 {2}	3.82634 {7}	2.55792 {5}	3.78472 {6}	1.44166 {1}	2.07292 {3}	2.07878 {4}	4.80631 {8}
		$\hat{\lambda}$	0.36450 {7}	0.36155 {6}	0.30320 {4}	0.36118 {5}	0.23009 {1}	0.28421 {3}	0.26391 {2}	0.37564 {8}
		$\hat{\rho}$	0.54698 {4}	0.78215 {7}	0.61909 {5}	0.79939 {8}	0.50309 {1}	0.51758 {2}	0.53353 {3}	0.62378 {6}
	BIAS	$\hat{\eta}$	1.00069 {2}	1.47422 {6}	1.19200 {5}	1.49256 {7}	0.88202 {1}	1.02609 {3}	1.05025 {4}	1.56611 {8}
		$\sum Ranks$	36 {4}	58 {6}	41 {5}	59 {7}	11 {1}	27 {3}	26 {2}	66 {8}
		$\hat{\lambda}$	0.14758 {8}	0.13872 {6}	0.10844 {4}	0.14579 {7}	0.07759 {1}	0.09349 {2}	0.09838 {3}	0.13697 {5}
150	MSE	$\hat{\rho}$	0.92239 {2}	1.43070 {7}	1.12845 {6}	1.49678 {8}	0.85192 {1}	0.93250 {3}	0.99038 {4}	1.10125 {5}
		$\hat{\eta}$	0.50255 {2}	0.83233 {6}	0.64532 {5}	0.88198 {8}	0.43769 {1}	0.52630 {3}	0.57960 {4}	0.84093 {7}
		$\hat{\lambda}$	0.05856 {5}	0.06133 {6}	0.04450 {4}	0.06696 {7}	0.01878 {1}	0.03541 {3}	0.03195 {2}	0.07052 {8}
	MRE	$\hat{\rho}$	1.91540 {4}	4.57570 {7}	2.67697 {6}	5.10395 {8}	1.61120 {1}	1.69301 {2}	1.89592 {3}	2.53115 {5}
		$\hat{\eta}$	0.82709 {2}	2.26556 {6}	1.51597 {5}	2.50621 {7}	0.64108 {1}	0.99426 {3}	1.16064 {4}	2.76619 {8}
		$\hat{\lambda}$	0.29515 {8}	0.27745 {6}	0.21688 {4}	0.29157 {7}	0.15518 {1}	0.18698 {2}	0.19677 {3}	0.27394 {5}
	BIAS	$\hat{\rho}$	0.36896 {2}	0.57228 {7}	0.45138 {6}	0.59871 {8}	0.34077 {1}	0.37300 {3}	0.39615 {4}	0.44050 {5}
		$\hat{\eta}$	0.67007 {2}	1.10978 {6}	0.86043 {5}	1.17597 {8}	0.58358 {1}	0.70174 {3}	0.77280 {4}	1.12124 {7}
		$\sum Ranks$	35 {4}	57 {7}	45 {5}	68 {8}	9 {1}	24 {2}	31 {3}	55 {6}
500	MSE	$\hat{\lambda}$	0.12309 {8}	0.08793 {6}	0.06391 {4}	0.09260 {7}	0.05036 {1}	0.05535 {2}	0.06059 {3}	0.07905 {5}
		$\hat{\rho}$	0.61143 {2}	0.95673 {7}	0.75757 {6}	0.97509 {8}	0.55917 {1}	0.63515 {3}	0.68286 {4}	0.74182 {5}
		$\hat{\eta}$	0.32159 {3}	0.53020 {7}	0.37399 {5}	0.56033 {8}	0.28431 {1}	0.31824 {2}	0.35270 {4}	0.47723 {6}
	MRE	$\hat{\lambda}$	0.04606 {8}	0.02774 {6}	0.01201 {4}	0.02937 {7}	0.00679 {1}	0.00864 {2}	0.01079 {3}	0.02400 {5}
		$\hat{\rho}$	0.67820 {2}	1.82750 {7}	1.06885 {6}	1.89490 {8}	0.61590 {1}	0.71279 {3}	0.80921 {4}	0.95780 {5}
		$\hat{\eta}$	0.26872 {2}	0.99952 {7}	0.42299 {5}	1.09192 {8}	0.23541 {1}	0.28778 {3}	0.37266 {4}	0.89857 {6}
	BIAS	$\hat{\lambda}$	0.24618 {8}	0.17585 {6}	0.12782 {4}	0.18521 {7}	0.10073 {1}	0.11071 {2}	0.12119 {3}	0.15810 {5}
		$\hat{\rho}$	0.24457 {2}	0.38269 {7}	0.30303 {6}	0.39004 {8}	0.22367 {1}	0.25406 {3}	0.27314 {4}	0.29673 {5}
		$\hat{\eta}$	0.42878 {3}	0.70693 {7}	0.49866 {5}	0.74711 {8}	0.37908 {1}	0.42432 {2}	0.47027 {4}	0.63630 {6}
	MSE	$\sum Ranks$	38 {4}	60 {7}	45 {5}	69 {8}	9 {1}	22 {2}	33 {3}	48 {6}
		$\hat{\lambda}$	0.10168 {8}	0.03656 {7}	0.02776 {4}	0.03612 {6}	0.02199 {1}	0.02586 {2}	0.02773 {3}	0.03228 {5}
		$\hat{\rho}$	0.33642 {3}	0.48406 {7}	0.37327 {5}	0.48483 {8}	0.24964 {1}	0.32773 {2}	0.36691 {4}	0.39019 {6}
	MRE	$\hat{\eta}$	0.18002 {5}	0.21910 {8}	0.16020 {4}	0.21735 {7}	0.12224 {1}	0.14766 {2}	0.15973 {3}	0.19014 {6}
		$\hat{\lambda}$	0.04151 {8}	0.00314 {7}	0.00139 {3}	0.00293 {6}	0.00094 {1}	0.00115 {2}	0.00145 {4}	0.00217 {5}
		$\hat{\rho}$	0.17000 {2}	0.38170 {7}	0.22755 {5}	0.38663 {8}	0.13380 {1}	0.17333 {3}	0.21975 {4}	0.24821 {6}
	BIAS	$\hat{\eta}$	0.06131 {5}	0.11519 {8}	0.04724 {3}	0.10967 {7}	0.03052 {1}	0.03806 {2}	0.04894 {4}	0.07909 {6}
		$\hat{\lambda}$	0.20336 {8}	0.07312 {7}	0.05551 {4}	0.07225 {6}	0.04398 {1}	0.05173 {2}	0.05545 {3}	0.06457 {5}
		$\hat{\rho}$	0.13457 {3}	0.19362 {7}	0.14931 {5}	0.19393 {8}	0.09985 {1}	0.13109 {2}	0.14676 {4}	0.15607 {6}
	MSE	$\hat{\eta}$	0.24003 {5}	0.29213 {8}	0.21360 {4}	0.28980 {7}	0.16298 {1}	0.19688 {2}	0.21298 {3}	0.25352 {6}
		$\sum Ranks$	47 {5}	66 {8}	37 {4}	63 {7}	9 {1}	19 {2}	32 {3}	51 {6}

Table 5: Simulation results for $\phi = (\lambda = 0.5, \rho = 3.25, \eta = 0.75)^T$.

<i>n</i>	Est.	Est. Par.	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs
20	BIAS	$\hat{\lambda}$	0.22345 {4}	0.25171 {6}	0.23120 {5}	0.25816 {8}	0.17138 {1}	0.18484 {2}	0.20235 {3}	0.25177 {7}
		$\hat{\rho}$	3.72512 {4}	4.26633 {6}	3.79067 {5}	4.71241 {8}	3.61792 {3}	3.41229 {2}	3.28861 {1}	4.45377 {7}
		$\hat{\eta}$	2.05045 {3}	2.77149 {6}	2.58887 {5}	3.26678 {8}	1.60023 {1}	1.77584 {2}	2.19654 {4}	3.22501 {7}
	MSE	$\hat{\lambda}$	0.13609 {1}	0.18261 {6}	0.18005 {5}	0.19336 {8}	0.14371 {3}	0.13915 {2}	0.14687 {4}	0.19061 {7}
		$\hat{\rho}$	34.87300 {5}	41.23382 {6}	34.27229 {3}	53.03521 {7}	34.66815 {4}	23.83024 {1}	26.07850 {2}	56.51849 {8}
		$\hat{\eta}$	25.25914 {3}	34.97243 {5}	36.91643 {6}	57.90343 {8}	11.53254 {1}	15.64428 {2}	26.32366 {4}	54.62050 {7}
50	MRE	$\hat{\lambda}$	0.44691 {4}	0.50342 {6}	0.46239 {5}	0.51632 {8}	0.34275 {1}	0.36969 {2}	0.40471 {3}	0.50353 {7}
		$\hat{\rho}$	1.06432 {4}	1.21895 {6}	1.08305 {5}	1.34640 {8}	1.03369 {3}	0.97494 {2}	0.93960 {1}	1.27250 {7}
		$\hat{\eta}$	2.73394 {3}	3.69532 {6}	3.45182 {5}	4.35570 {8}	2.13364 {1}	2.36779 {2}	2.92873 {4}	4.30002 {7}
	BIAS	$\sum Ranks$	31 {4}	53 {6}	44 {5}	71 {8}	18 {2}	17 {1}	26 {3}	64 {7}
		$\hat{\lambda}$	0.14191 {6}	0.14230 {7}	0.10958 {4}	0.14655 {8}	0.07862 {1}	0.08496 {2}	0.09397 {3}	0.13672 {5}
		$\hat{\rho}$	1.73503 {3}	2.56374 {7}	2.04372 {6}	2.65694 {8}	1.64928 {1}	1.76118 {4}	1.68707 {2}	1.99777 {5}
80	MSE	$\hat{\eta}$	0.71730 {2}	1.37714 {6}	1.03616 {5}	1.50474 {8}	0.68748 {1}	0.74171 {3}	0.86026 {4}	1.46997 {7}
		$\hat{\lambda}$	0.05899 {5}	0.08064 {7}	0.05423 {4}	0.08383 {8}	0.02303 {1}	0.03510 {2}	0.03878 {3}	0.07440 {6}
		$\hat{\rho}$	7.89667 {4}	15.43483 {7}	9.80317 {5}	17.09932 {8}	6.72138 {3}	6.39467 {2}	6.05626 {1}	9.94687 {6}
	MRE	$\hat{\eta}$	2.89812 {2}	9.02721 {6}	5.89808 {5}	11.08838 {7}	4.24526 {1}	3.12867 {3}	3.92267 {4}	11.62145 {8}
		$\hat{\lambda}$	0.28383 {6}	0.28459 {7}	0.21917 {4}	0.29310 {8}	0.15724 {1}	0.16991 {2}	0.18794 {3}	0.27344 {5}
		$\hat{\rho}$	0.49572 {3}	0.73250 {7}	0.58392 {6}	0.75913 {8}	0.47122 {1}	0.50320 {4}	0.48202 {2}	0.57079 {5}
	BIAS	$\hat{\eta}$	0.95641 {2}	1.83618 {6}	1.38155 {5}	2.00632 {8}	0.91664 {1}	0.98895 {3}	1.14701 {4}	1.95996 {7}
		$\sum Ranks$	33 {4}	60 {7}	44 {5}	71 {8}	11 {1}	25 {2}	26 {3}	54 {6}
		$\hat{\lambda}$	0.11956 {8}	0.09887 {7}	0.07037 {4}	0.09634 {6}	0.05122 {1}	0.05161 {2}	0.06148 {3}	0.08821 {5}
150	MSE	$\hat{\rho}$	1.15355 {2}	1.88620 {7}	1.46691 {6}	1.93298 {8}	1.08006 {1}	1.25184 {4}	1.24395 {3}	1.42776 {5}
		$\hat{\eta}$	0.44680 {3}	0.93144 {7}	0.64397 {5}	0.93155 {8}	0.42762 {1}	0.43401 {2}	0.54428 {4}	0.88923 {6}
		$\hat{\lambda}$	0.04560 {8}	0.04192 {7}	0.02194 {4}	0.04031 {6}	0.00853 {1}	0.00950 {2}	0.01555 {3}	0.03598 {5}
	MRE	$\hat{\rho}$	3.08772 {3}	8.04814 {7}	4.56504 {6}	8.82459 {8}	2.49439 {1}	3.22686 {4}	3.01696 {2}	4.38758 {5}
		$\hat{\eta}$	0.78977 {3}	4.45864 {6}	2.45015 {5}	4.71270 {7}	0.78722 {2}	0.77506 {1}	1.47251 {4}	4.86193 {8}
		$\hat{\lambda}$	0.23912 {8}	0.19775 {7}	0.14075 {4}	0.19268 {6}	0.10245 {1}	0.10321 {2}	0.12297 {3}	0.17643 {5}
	BIAS	$\hat{\rho}$	0.32959 {2}	0.53891 {7}	0.41912 {6}	0.55228 {8}	0.30859 {1}	0.35767 {4}	0.35541 {3}	0.40793 {5}
		$\hat{\eta}$	0.59574 {3}	1.24192 {7}	0.85863 {5}	1.24207 {8}	0.57016 {1}	0.57867 {2}	0.72571 {4}	1.18564 {6}
		$\sum Ranks$	40 {4}	62 {7}	45 {5}	65 {8}	10 {1}	23 {2}	29 {3}	50 {6}
500	MSE	$\hat{\lambda}$	0.10414 {8}	0.05543 {6}	0.03862 {4}	0.05824 {7}	0.03161 {1}	0.03394 {2}	0.03829 {3}	0.04708 {5}
		$\hat{\rho}$	0.75373 {2}	1.25077 {7}	0.95256 {5}	1.26414 {8}	0.70173 {1}	0.85530 {3}	0.87532 {4}	0.95621 {6}
		$\hat{\eta}$	0.29188 {3}	0.49830 {7}	0.32719 {5}	0.53283 {8}	0.25281 {1}	0.27368 {2}	0.31964 {4}	0.42561 {6}
	MRE	$\hat{\lambda}$	0.04020 {8}	0.01201 {6}	0.00383 {4}	0.01453 {7}	0.00220 {1}	0.00231 {2}	0.00372 {3}	0.00741 {5}
		$\hat{\rho}$	1.06960 {2}	3.16105 {7}	1.71546 {6}	3.22460 {8}	0.99003 {1}	1.33004 {4}	1.29901 {3}	1.62141 {5}
		$\hat{\eta}$	0.17893 {3}	1.22618 {7}	0.33105 {5}	1.43765 {8}	0.16070 {1}	0.16901 {2}	0.30099 {4}	0.84337 {6}
	BIAS	$\hat{\lambda}$	0.20827 {8}	0.11087 {6}	0.07725 {4}	0.11649 {7}	0.06322 {1}	0.06788 {2}	0.07659 {3}	0.09416 {5}
		$\hat{\rho}$	0.21535 {2}	0.35736 {7}	0.27216 {5}	0.36118 {8}	0.20049 {1}	0.24437 {3}	0.25009 {4}	0.27320 {6}
		$\hat{\eta}$	0.38917 {3}	0.66441 {7}	0.43626 {5}	0.71044 {8}	0.33708 {1}	0.36491 {2}	0.42619 {4}	0.56748 {6}
	MRE	$\sum Ranks$	39 {4}	60 {7}	43 {5}	69 {8}	9 {1}	22 {2}	32 {3}	50 {6}
		$\hat{\lambda}$	0.09197 {8}	0.02357 {7}	0.01856 {4}	0.02353 {6}	0.01414 {1}	0.01682 {2}	0.01849 {3}	0.02125 {5}
		$\hat{\rho}$	0.39850 {2}	0.62366 {7}	0.47666 {5}	0.62447 {8}	0.28841 {1}	0.42463 {3}	0.47298 {4}	0.50589 {6}
	BIAS	$\hat{\eta}$	0.17708 {6}	0.20000 {8}	0.14954 {4}	0.19936 {7}	0.10819 {1}	0.13262 {2}	0.14919 {3}	0.17529 {5}
		$\hat{\lambda}$	0.03960 {8}	0.00106 {7}	0.00059 {4}	0.00102 {6}	0.00039 {1}	0.00047 {2}	0.00058 {3}	0.00081 {5}
		$\hat{\rho}$	0.25115 {2}	0.64240 {7}	0.37638 {5}	0.65756 {8}	0.21513 {1}	0.29547 {3}	0.35918 {4}	0.41858 {6}
	MSE	$\hat{\eta}$	0.06041 {6}	0.08146 {8}	0.03973 {4}	0.07568 {7}	0.02510 {1}	0.02989 {2}	0.03842 {3}	0.05833 {5}
		$\hat{\lambda}$	0.18394 {8}	0.04714 {7}	0.03713 {4}	0.04706 {6}	0.02828 {1}	0.03364 {2}	0.03698 {3}	0.04249 {5}
		$\hat{\rho}$	0.11386 {2}	0.17819 {7}	0.13619 {5}	0.17842 {8}	0.08240 {1}	0.12132 {3}	0.13514 {4}	0.14454 {6}
	BIAS	$\hat{\eta}$	0.23610 {6}	0.26666 {8}	0.19938 {4}	0.26582 {7}	0.14425 {1}	0.17683 {2}	0.19892 {3}	0.23372 {5}
		$\sum Ranks$	48 {5.5}	66 {8}	39 {4}	63 {7}	9 {1}	21 {2}	30 {3}	48 {5.5}

Table 6: Partial and overall ranks of all estimation methods for various combinations of ϕ .

λ	ρ	η	ϕ	n	MLEs	LSEs	WLSEs	CRVMEs	MPSEs	PCEs	ADEs	RADEs			
0.5	0.35	0.75	0.75	20	3	7	5	6	4	8	1	2			
				50	1	7	5	6	4	8	2	3			
				80	1	7	4	6	5	8	3	2			
				150	1	7	4	6	5	8	2	3			
				500	5	7	4	6	1	8	3	2			
0.5	1.25	0.75	0.75	20	7.5	5	2.5	7.5	1	2.5	4	6			
				50	8	5	3	7	1	2	4	6			
				80	7	6	5	8	1	2	3	4			
				150	7	6	3	8	1	2	4.5	4.5			
				500	4	8	2	7	1	6	3	5			
0.5	2.25	0.75	0.75	20	4	5	6	7.5	1	2	3	7.5			
				50	4	6	5	7	1	3	2	8			
				80	4	7	5	8	1	2	3	6			
				150	4	7	5	8	1	2	3	6			
				500	5	8	4	7	1	2	3	6			
0.5	3.25	0.75	0.75	20	4	6	5	8	2	1	3	7			
				50	4	7	5	8	1	2	3	6			
				80	4	7	5	8	1	2	3	6			
				150	4	7	5	8	1	2	3	6			
				500	5.5	8	4	7	1	2	3	5.5			
0.45	0.35	0.75	0.75	20	3	7	4	5.5	5.5	8	2	1			
				50	1	7	6	4	5	8	3	2			
				80	2	7	4	5.5	5.5	8	1	3			
				150	1	7	4	6	5	8	2	3			
				500	6	7	4	5	2	8	3	1			
0.45	1.25	0.75	0.75	20	8	5	3	7	1	2	4	6			
				50	7	5	4	8	1	2	3	6			
				80	7	6	4	8	1	2	3	5			
				150	7	6	5	8	1	4	3	2			
				500	4	7	3	8	1	6	2	5			
0.45	1.25	1.25	1.25	20	7	5	4	8	1	2.5	2.5	6			
				50	7	5	4	6	1	3	2	8			
				80	6	5	2.5	7	1	4	2.5	8			
				150	5	6	3	8	1	4	2	7			
				500	2	7	4	8	1	5	3	6			
0.45	2.25	0.75	0.75	20	5	6	4	7	1	3	2	8			
				50	4	6	5	7	1	2	3	8			
				80	4	6	5	8	1	2	3	7			
				150	4	8	5	7	1	2	3	6			
				500	5	7	4	8	1	2	3	6			
0.45	2.25	1.25	1.25	20	4	6	5	8	3	1	2	7			
				50	4	6	5	8	1	2	3	7			
				80	3	6	5	8	1	2	4	7			
				150	3	7	5	8	1	2	4	6			
				500	4.5	8	4.5	7	1	2	3	6			
0.45	2.25	2.5	2.5	20	5	6	4	8	1.5	1.5	3	7			
				50	3	6.5	5	8	1	2	4	6.5			
				80	2	7	5	8	1	3	4	6			
				150	2	8	5	7	1	3	4	6			
				500	2	8	5	7	1	3	4	6			
3	0.35	0.75	0.75	20	2	7	5	6	3	8	1	4			
				50	1	7	5	6	4	8	3	2			
				80	1	7	5	6	3	8	2	4			
				150	1	7	4	6	3	8	2	5			
				500	2	7	4.5	6	1	8	3	4.5			
3	0.35	1.25	1.25	20	1	6	5	7	3	8	2	4			
				50	1	7	3	6	4	8	2	5			
				80	1	7	3	6	4	8	2	5			
				150	1	6	2	7	4	8	3	5			
				500	3	7	2	6	1	8	4	5			
$\sum Ranks$				228.5	394.5	255	422	115.5	261.5	169.5	313.5				
Overall Rank				3	7	4	8	1	5	2	6				

Tables 7 and 8 list the MLEs of the fitted distributions with their standard errors (SEs) and the values of goodness-of-fit measures. Based on Tables 7 and 8, it is noted that the ERKiEx distribution has shown great flexibility and high efficiency in modeling failure times and time-to-failure data, respectively. These results show that the new distribution is the best competing distribution and that it is a good candidate for statistical modeling for real-life data in the fields of engineering, insurance and other applied fields.

The plots of the fitted densities, CDFs, SFs and probability-probability (PP) of the ERKiEx

distribution for both data sets are displayed in Figures 3 and 4. These plots supports the superiority of the ERKiEx distribution over other competing models in fitting the two analyzed data.

Table 7: Goodness-of-fit measures, MLEs and their SEs for first data

Model	Param.	Estimates	(SEs)	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	PV
ERKiEx	$\hat{\lambda}$	0.30407	(0.05002)								
	$\hat{\rho}$	2.57671	(1.00282)	108.315	108.6579	115.2272	111.0724	0.02695	0.21123	0.05793	0.96502
	$\hat{\eta}$	2.38887	(1.68803)								
RKiEx	$\hat{\lambda}$	5.14301	(0.37121)								
	$\hat{\rho}$	5.82568	(0.00548)	108.42110	108.59010	113.02920	110.25930	0.03588	0.31938	0.07532	0.79517
	$\hat{\eta}$	8091.32581	(7449.52828)								
MOEx	$\hat{\lambda}$	3.61778	(0.35827)	107.89810	108.06710	112.50620	109.73630	0.04433	0.27236	0.05981	0.95388
	$\hat{\rho}$	9.12743	(4.74994)								
	$\hat{\eta}$	129.19923	(463.07494)	108.2628	108.6057	115.175	111.0202	0.02677	0.21033	0.05749	0.96732
KwEx	$\hat{\lambda}$	12.95957	(0.54411)								
	$\hat{\rho}$	0.15592	(0.02687)								
	$\hat{\eta}$	24.22772	(3.95590)	110.33000	110.49900	114.93810	112.16820	0.08711	0.57184	0.06814	0.88207
GEx	$\hat{\lambda}$	9.78001	(1.61349)								
	$\hat{\rho}$	2.01915	(0.17163)	121.60650	121.77560	126.21470	123.44480	0.21723	1.40531	0.09532	0.51206
	$\hat{\eta}$	89.43226	(32.47473)								
Ex	$\hat{\lambda}$	0.4036709	0.04692	284.2593	284.3148	286.5633	285.1784	0.08758591	0.5749472	0.4494711	2.06945

Table 8: Goodness-of-fit measures, MLEs and their SEs for second data

Model	Param.	Estimates	(SEs)	AIC	CAIC	BIC	HQIC	W^*	A^*	KS	PV
ERKiEx	$\hat{\lambda}$	0.08069	(0.00178)								
	$\hat{\rho}$	12.95957	(0.54411)	162.16740	162.83410	167.23400	163.99930	0.01552	0.10302	0.05645	0.99956
	$\hat{\eta}$	0.15592	(0.02687)								
RKiEx	$\hat{\lambda}$	2.92823	(0.39777)								
	$\hat{\rho}$	0.09936	(0.00401)	166.53870	166.86300	169.91640	167.76000	0.05347	0.41126	0.10239	0.79581
	$\hat{\eta}$	285.76186	(240.44551)								
MOEx	$\hat{\lambda}$	0.88373	(0.12032)	171.20520	171.52950	174.58290	172.42640	0.08601	0.61795	0.09128	0.89282
	$\hat{\rho}$	8.25043	(2.63190)								
	$\hat{\eta}$	1.50000	(0.71842)	183.66580	184.33240	188.73240	185.49770	0.24110	1.56617	0.13919	0.42048
KwEx	$\hat{\lambda}$	7.22272	(1.69085)								
	$\hat{\rho}$	1.323514	(0.27941)	178.82050	179.14490	182.19830	180.04180	0.20526	1.36163	0.12776	0.53110
	$\hat{\eta}$	9.51458	(2.89609)								
Ex	$\hat{\lambda}$	0.1599364	0.02528718	228.6385	228.7438	230.3274	229.2492	0.2065203	1.368904	0.3631084	5.25014

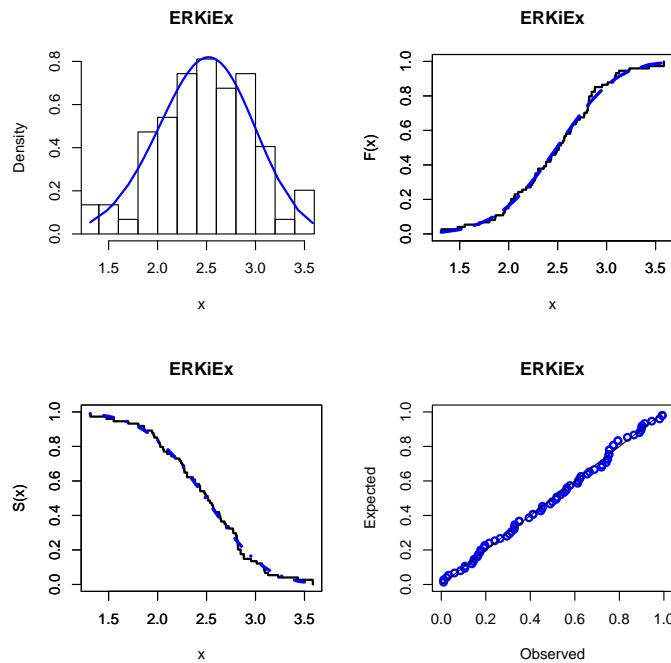
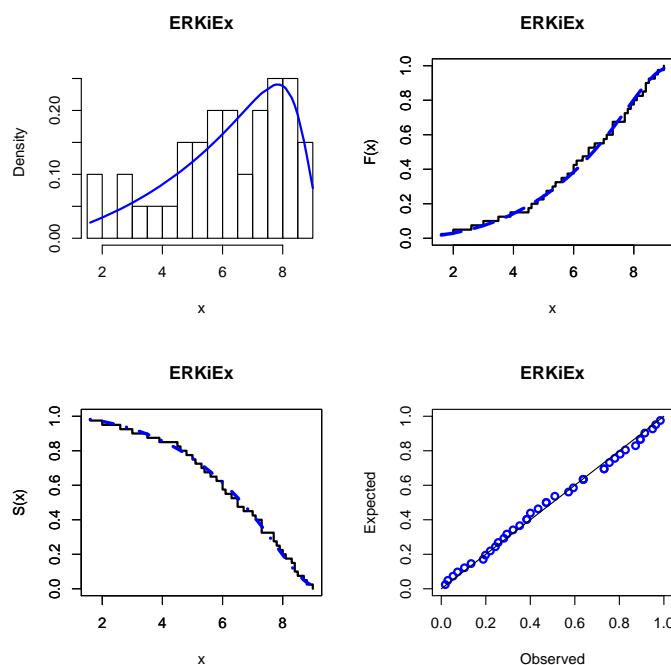
Also, different estimation methods are adopted in estimating the parameters of competing models from the two data sets. Tables 9 and 10 provide the ERKiEx estimates under various estimation methods and some other selection measures for the two data sets, respectively. Table 9 shows that all the classical methods are suitable in statistical modeling processes and they all perform well, because the results of each of them are convergence in a way that makes it difficult to fully weight them.

Table 9: The estimates under various estimation methods KS and PV for first data

Methods	$\hat{\lambda}$	$\hat{\rho}$	$\hat{\eta}$	W^*	A^*	KS	PV
WLSEs	0.2962966	2.7865780	1.9482550	0.0174775	0.1369411	0.0412596	0.9996449
LSEs	0.2883174	3.0167530	1.7028590	0.0179250	0.1421064	0.0417199	0.9995674
MLEs	0.3040700	2.5767100	2.3888700	0.0269500	0.2112300	0.0579300	0.9650200
MPSEs	0.3086349	2.4425660	2.3045840	0.0184185	0.1402535	0.0489545	0.9951910
CRVMEs	0.2887883	3.0643980	1.7236630	0.0180746	0.1433939	0.0395884	0.9998366
ADEs	0.2992816	2.7566580	2.0543600	0.0174917	0.1365762	0.0393512	0.9998548
RADEs	0.3107710	2.5265120	2.4523860	0.0182241	0.1390649	0.0430040	0.9992738
PCEs	0.2992816	2.7566580	2.0543600	0.0174917	0.1365762	0.0393512	0.9998548

Table 10: The estimates under various estimation methods, the KS and PV for second data

Methods	$\hat{\lambda}$	$\hat{\rho}$	$\hat{\eta}$	W^*	A^*	KS	PV
WLSEs	0.2962966	2.7865780	1.9482550	0.0174775	0.1369411	0.0412596	0.9996449
LSEs	0.2883174	3.0167530	1.7028590	0.0179250	0.1421064	0.0417199	0.9995674
MLEs	0.0806900	12.9595700	0.1559200	0.0155200	0.1030200	0.0564500	0.9995600
MPSEs	0.3086349	2.4425660	2.3045840	0.0184185	0.1402535	0.0489545	0.9951910
CRVMEs	0.2887883	3.0643980	1.7236630	0.0180746	0.1433939	0.0395884	0.9998366
ADEs	0.2992816	2.7566580	2.0543600	0.0174917	0.1365762	0.0393512	0.9998548
RADEs	0.3107710	2.5265120	2.4523860	0.0182241	0.1390649	0.0430040	0.9992738
PCEs	0.2992816	2.7566580	2.0543600	0.0174917	0.1365762	0.0393512	0.9998548

**Figure 3:** The fitted ERKiEx PDF, CDF, SF, and PP plots for first data.**Figure 4:** The fitted ERKiEx PDF, CDF, SF, and PP plots for second data.

6 Conclusions

In this paper, a new class called the exponentiated reduced-Kies-G (ERKi-G) family is presented. The key mathematical properties of the ERKi-G family are explored. A sub-model of the ERK-G family called the ERKi-exponential (ERKiEx) distribution is introduced and studied theoretically and practically. The density of the ERKiEx model provides many useful shapes including right skewed, left skewed, and symmetric. The failure rate of the ERKiEx model can be monotonically decreasing, bathtub, monotonically increasing and J-shaped. Eight estimation methods are adopted to estimate ERKiEx parameters. Comprehensive set of simulation studies to evaluate the behavior of the eight estimation methods are presented. Partial and overall ranks are calculated for all simulation results to rank the proposed estimators with respect to their absolute biases, mean relative errors and mean square errors. Based on our study, the maximum product of spacing and Anderson–Darling methods are recommended to estimate the ERKiEx parameters. Two real-life data sets are fitted using the ERKiEx distribution and other models. The two data show that the proposed ERKiEx distribution provides better fit as compared to other competing models.

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